V. *An Appendix to the Paper in the Philoſophical Tranſactions for the Year* 1778, *Number* XLII, *pages* 902 *et ſeq. intitled,* "*A Method of extending* Cardan's "*Rule for reſolving one Caſe of the Cubick Equation* "$x^3-qx=r$ *to the other Caſe of the ſame Equation,* "*which it is not naturally fitted to ſolve, and which is* "*therefore called* the irreducible Caſe." *By* Francis Maſeres, *Eſq. F. R. S. Curſitor Baron of the Exchequer.*

Read Nov. 4, 1779.

ARTICLE I.

IN the above-mentioned paper in the Philoſophical Tranſactions the expreſſion $\sqrt[3]{e}\times$ the infinite ſeries $2+\frac{2ss}{9ee}-\frac{20s^4}{243e^4}+\frac{308s^6}{6561e^6}-$ &c. is ſhewn to be equal to the root of the equation $x^3-qx=r$, whenever $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, but greater than one half of it, or than $\frac{q^3}{54}$. This expreſſion is wholly tranſcendental, or compoſed of an infinite number of terms, to wit, the terms of the ſeries $2+\frac{2ss}{9ee}-\frac{20s^4}{243e^4}+\frac{308s^6}{6561e^6}-$ &c. multiplied into the cube-root of e. But I have ſince thought that it might be convenient on ſome occaſions to divide this expreſſion, if poſſible, into two others, whereof the one ſhould be a mere algebraick expreſſion, or conſiſt of a finite number of terms, and the other ſhould be tranſcendental, or involve in it an infinite ſeries. And I have accordingly diſcovered a method of doing this, which I will now proceed to deſcribe.

ART. 2. In the above-mentioned paper in the Philoſophical Tranſactions I denoted the exceſs of $\frac{q^3}{27}$ above $\frac{rr}{4}$ in the ſecond caſe of the equation $x^3 - qx = r$, as well as the exceſs of $\frac{rr}{4}$ above $\frac{q^3}{27}$ in the firſt caſe of it, by the letters ss. But I have ſince thought that it might have been better to denote the exceſs of $\frac{q^3}{27}$ above $\frac{rr}{4}$ in the ſecond caſe of that equation by the letters zz, in order the more clearly to diſtinguiſh it from the oppoſite difference $\frac{rr}{4} - \frac{q^3}{27}$ in the firſt caſe of it, which was denoted by ss. And I therefore in the courſe of the following pages ſhall uſe the letters zz inſtead of ss to denote the ſaid exceſs of $\frac{q^3}{27}$ above $\frac{rr}{4}$ in the ſecond caſe of the ſaid equation, or the difference $\frac{q^3}{27} - \frac{rr}{4}$.

ART. 3. Now, if zz be ſubſtituted inſtead of ss in the expreſſion $\sqrt[3]{e}$ × the infinite ſeries $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6}$ − &c. that expreſſion will thereby be converted into the following expreſſion, to wit, $\sqrt[3]{e}$ × the infinite ſeries $2 + \frac{2zz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6}$ − &c. Therefore, if $\frac{rr}{4}$ be leſs than $\frac{q^3}{27}$ but greater than $\frac{q^3}{54}$, and e be put $= \frac{r}{2}$, and zz be put $= \frac{q^3}{27} - \frac{rr}{4}$, the root of the equation $x^3 - qx = r$ will be equal to $\sqrt[3]{e} \times$ the infinite ſeries $2 + \frac{2zz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} -$ &c.

ART. 4. The numeral coefficients $\frac{2}{9}, \frac{20}{243}, \frac{308}{6561}$, &c. of $\frac{zz}{ee}, \frac{z^4}{e^4}, \frac{z^6}{e^6}$, &c. in this ſeries are exactly double of $\frac{1}{9}, \frac{10}{243}, \frac{154}{6561}$, &c. which are the numeral coefficients of the ſame powers of the fraction $\frac{z}{e}$ in the ſeries $1 + \frac{z}{3e} - \frac{zz}{9ee} + \frac{5z^3}{81e^3} - \frac{10z^4}{243e^4} + \frac{22z^5}{729e^5} - \frac{154z^6}{6561e^6} + \frac{2618z^7}{137,781e^7} -$ &c. which is equal to the cube-root of the binomial quantity $1 + \frac{z}{e}$; or, if the numeral coefficients of the ſaid latter ſeries be denoted by the capital letters A, B, C, D, E, F, G, H, &c. reſpectively, ſo that A ſhall be $= 1$, B $= \frac{1}{3}$, C $= \frac{1}{9}$, D $= \frac{5}{81}$, E $= \frac{10}{243}$, F $= \frac{22}{729}$, G $= \frac{154}{6561}$, and H $= \frac{2618}{137,781}$, and ſo on, the ſaid numeral coefficients $\frac{2}{9}, \frac{20}{243}, \frac{308}{6561}$, &c. will be equal to 2C, 2E, 2G, &c. and the ſeries mentioned in the laſt Article will be $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c. and conſequently the root of the equation $x^3 - qx = r$, in the ſecond caſe of it, in which $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, will be equal to the expreſſion $\sqrt[3]{e} \times$ the ſeries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c.

ART. 5. Now the ſeries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c.

is equal to the ſum of the two following ſerieſes, to wit, $2A - \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} - \frac{2Gz^6}{e^6} -$ &c. (in which all the terms following the firſt term are marked with the ſign $-$, or are ſubtracted from the firſt term), and $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} +$ &c; and the ſeries $2A - \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} - \frac{2Gz^6}{e^6} -$ &c. is equal to the ſum of the two ſerieſes $A + \frac{Bz}{e} - \frac{Czz}{ee} + \frac{Dz^3}{e^3} - \frac{Ez^4}{e^4} + \frac{Fz^5}{e^5} - \frac{Gz^6}{e^6} +$ &c. and $A - \frac{Bz}{e} - \frac{Czz}{ee} - \frac{Dz^3}{e^3} - \frac{Ez^4}{e^4} - \frac{Fz^5}{e^5} - \frac{Gz^6}{e^6} -$ &c. which are reſpectively equal to the cube-roots of the binomial quantities $1 + \frac{z}{e}$ and $1 - \frac{z}{e}$. Therefore the ſeries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c. is $= \sqrt{}^3\,\overline{1 + \frac{z}{e}} + \sqrt{}^3\,\overline{1 - \frac{z}{e}} +$ the infinite ſeries $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} +$ &c. Conſequently the expreſſion $\sqrt{}^3 e \times$ the ſeries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c. is $= \sqrt{}^3 e \times \sqrt{}^3\,\overline{1 + \frac{z}{e}} + \sqrt{}^3 e \times \sqrt{}^3\,\overline{1 - \frac{z}{e}} + \sqrt{}^3 e \times$ the infinite ſeries $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} +$ &c. $= \sqrt{}^3\,\overline{e + z} + \sqrt{}^3\,\overline{e - z} + \sqrt{}^3 e \times$ the infinite ſeries $\frac{4Czz}{ee} + \frac{4Gz^6}{e^6} +$ &c. $= \sqrt{}^3\,\overline{e + z} + \sqrt{}^3 e - z + 4\sqrt{}^3 e \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} +$ &c. Therefore the root of the equation $x^3 - qx = r$, in the ſecond caſe of it, in which $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, is equal to $\sqrt{}^3\,\overline{e + z} + \sqrt{}^3\,\overline{e - z} + 4\sqrt{}^3 e \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{14}}{e^{14}} +$ &c; of which expreſſion the firſt part, to wit, $\sqrt[3]{\overline{e+z}} + \sqrt[3]{\overline{e-z}}$ is algebraick, and the latter part, to wit, $\sqrt[3]{e} \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} +$ &c. is tranſcendental. Q. E. I.(a)

Of the convergency of the Series obtained in the preceding Article.

ART. 6. This ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} +$ &c. evidently converges faſter than the ſeries $2A + \frac{2Czz}{ee} - \frac{2Ez^4}{e^4} + \frac{2Gz^6}{e^6} -$ &c. or $2 + \frac{2zz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} -$ &c: and conſequently the expreſſion $\sqrt[3]{\overline{e+z}} + \sqrt[3]{\overline{e-z}} + 4\sqrt[3]{e} \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} +$ &c. ſeems rather fitter to exhibit the value of x in the equation $x^3 - qx = r$, in the ſecond caſe of it (in which $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$), to a conſiderable degree of exactneſs than the former expreſſion $\sqrt{{}^3 e} \times$ the ſeries $2 + \frac{2zz}{9ee} - \frac{20z^4}{243e^4} + \frac{308z^6}{6561e^6} - $ &c.

(a) N. B. I have been informed that both this mixed expreſſion of the root of the equation $x^3 - qx = r$ in the ſecond caſe of it, and the merely tranſcendental expreſſion of it publiſhed in the former paper, and from which this expreſſion is derived, were invented by Monſieur NICOLE, and publiſhed in the memoirs of the French Academy of Sciences ſo long ago as the year 1738; and the latter of them, to wit, the tranſcendental expreſſion $\sqrt[3]{e} \times$ the ſeries $2 + \frac{2ss}{9ee} - \frac{20s^4}{243e^4} + \frac{308s^6}{6561e^6} -$ &c. I had myſelf ſeen many years ago in Monſieur CLAIRAUT's algebra, in the place cited in the 50th Article of my former paper, to wit, in pages 286, 287, 288. But it was obtained by the intervention of negative quantities, and the roots of negative quantities, which gave it, in my opinion, an air of great obſcurity. And therefore I thought an inveſtigation of the ſame ſeries, by a method that keeps clear of thoſe difficulties, might not be unacceptable to the lovers of theſe ſciences, nor unworthy of a place in the Tranſactions of this learned body.

A Computation of the four firſt Terms of the Series obtained in Art. 5.

ART. 7. The firſt fifteen terms of the infinite ſeries which is equal to the cube-root of $1 + \frac{z}{e}$ are as follows; to wit, $1 + \frac{z}{3e} - \frac{zz}{9ee} + \frac{5z^3}{81e^3} - \frac{10z^4}{243e^4} + \frac{22z^5}{729e^5} - \frac{154z^6}{6561e^6} + \frac{2618z^7}{137,781e^7} - \frac{935z^8}{59,049e^8} + \frac{21505z^9}{1,594,323e^9} - \frac{55913z^{10}}{4,782,969e^{10}} + \frac{147,407z^{11}}{14,348,907e^{11}} - \frac{1,179,256z^{12}}{129,140,163e^{12}} + \frac{3,174,920z^{13}}{387,420,489e^{13}} - \frac{60,323,480z^{14}}{8,135,830,269e^{14}} + $ &c; or, in decimal fractions, $1 +$
.333,333,333, &c. $\times \frac{z}{e}$ − .111,111,111, &c. $\times \frac{zz}{ee}$ +
.061,728,395, &c. $\times \frac{z^3}{e^3}$ − .041,152,263, &c. $\times \frac{z^4}{e^4}$ +
.030,178,326, &c. $\times \frac{z^5}{e^5}$ − .023,472,031, &c. $\times \frac{z^6}{e^6}$ +
.019,001,167, &c. $\times \frac{z^7}{e^7}$ − .015,834,305, &c. $\times \frac{z^8}{e^8}$ +
.013,488,482, &c. $\times \frac{z^9}{e^9}$ − .011,690,017, &c. $\times \frac{z^{10}}{e^{10}}$ +
.010,273,045, &c. $\times \frac{z^{11}}{e^{11}}$ − .009,131,595, &c. $\times \frac{z^{12}}{e^{12}}$ +
.008,195,021, &c. $\times \frac{z^{13}}{e^{13}}$ − .007,414,542, &c. $\times \frac{z^{14}}{e^{14}}$.
Therefore the four firſt terms of the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} +$

$\frac{Lz^{10}}{e^{10}}+\frac{Pz^{14}}{e^{14}}+\frac{Tz^{18}}{e^{18}}+$ &c. are $\frac{zz}{9ee}+\frac{154z^6}{6561e^6}+\frac{559\,13z^{10}}{4{,}782{,}969e^{10}}+\frac{60{,}323{,}480z^{14}}{8{,}135{,}830{,}269e^{14}}$, or, in decimal fractions, .111,111,111, &c. $\times\frac{zz}{ee}+$ .023,472,031, &c. $\times\frac{z^6}{e^6}+$ .011,690,017, &c. $\times\frac{z^{10}}{e^{0}}+$ .007,414,542, &c. $\times\frac{z^{14}}{e^{14}}$. Therefore the root of the cubick equation $x^3-qx=r$, in the ſecond caſe of it (in which $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$), is equal to $\sqrt[3]{e+z}+\sqrt[3]{e-z}+$ $4\sqrt[3]{e}\times$ the ſeries $\frac{zz}{9ee}+\frac{154z^6}{6561e^6}+\frac{55{,}913z^{10}}{4{,}782{,}969e^{10}}+\frac{60{,}323{,}480z^{14}}{8{,}135{,}830{,}269e^{14}}+$ &c. *ad infinitum*, or $\sqrt[3]{e+z}+\sqrt[3]{e-z}+4\sqrt[3]{e}\times$ the ſeries .111,111,111, &c. $\times\frac{zz}{ee}+$ .023,472,031, &c. $\times\frac{z^6}{e^6}+$ .011,690,017, &c. $\times\frac{z^{10}}{e^{0}}+$ .007,414,542, &c. $\times\frac{z^{14}}{e^{4}}+$ &c. *ad infinitum.*

Of the beſt Manner of Proceeding to the Computation of more Terms of the ſaid Series, if required.

ART. 8. If more than four terms of this laſt ſeries are required, it will be neceſſary to compute the ſeries $1+\frac{z}{3e}-\frac{zz}{9ee}+\frac{5z^3}{81e^3}-\frac{10z^4}{243e^4}+\frac{22z^5}{729e^5}-\frac{154z^6}{6561e^6}+$ &c.(which is $=\sqrt[3]{1+\frac{z}{e}}$), to more than fifteen terms; in order to which it will be convenient to expreſs the terms of that ſeries in the following manner, to wit, $1+\frac{1}{3}\times\frac{Az}{e}-\frac{2}{6}\times\frac{Bzz}{ee}+\frac{5}{9}\times\frac{Cz^3}{e^3}-\frac{8}{12}\times\frac{Dz^4}{e^4}+\frac{11}{15}\times\frac{Ez^5}{e^5}-\frac{14}{18}\times\frac{Fz^6}{e^6}+\frac{17}{21}\times\frac{Gz^7}{e^7}-\frac{20}{24}\times\frac{Hz^8}{e^8}+\frac{23}{27}\times\frac{Iz^9}{e^9}-\frac{26}{30}\times\frac{Kz^{10}}{e^{10}}+\frac{29}{33}\times$

$$\frac{Lz^{11}}{e^{11}} - \frac{32}{36} \times \frac{Mz^{12}}{e^{12}} + \frac{35}{39} \times \frac{Nz^{13}}{e^{13}} - \frac{38}{42} \times \frac{Oz^{14}}{e^{14}} + \frac{41}{45} \times \frac{Pz^{15}}{e^{15}} - \frac{44}{48} \times \frac{Qz^{16}}{e^{16}} + \frac{47}{51} \times \frac{Rz^{17}}{e^{17}} - \frac{50}{54} \times \frac{Sz^{18}}{e^{18}} + \frac{53}{57} \times \frac{Tz^{19}}{e^{19}} - \frac{56}{60} \times \frac{Vz^{20}}{e^{20}} + \frac{59}{63} \times \frac{Wz^{21}}{e^{21}} - \frac{62}{66} \times \frac{Xz^{22}}{e^{22}} + \&c.$$ or

$$A + \frac{Bz}{e} - \frac{Czz}{ee} + \frac{Dz^3}{e^3} - \frac{Ez^4}{e^4} + \frac{Fz^5}{e^5} - \frac{Gz^6}{e^6} + \frac{Hz^7}{e^7} - \frac{Iz^8}{e^8} + \frac{Kz^9}{e^9} - \frac{Lz^{10}}{e^{10}} + \frac{Mz^{11}}{e^{11}} - \frac{N^{12}}{e^{12}} + \frac{Oz^{13}}{e^{13}} - \frac{Pz^{14}}{e^{14}} + \frac{41}{45} \times \frac{Pz^{15}}{e^{15}} - \frac{44}{48} \times \frac{Qz^{16}}{e^{16}} + \frac{47}{51} \times \frac{Rz^{17}}{e^{17}} - \frac{50}{54} \times \frac{Sz^{18}}{e^{18}} + \frac{53}{57} \times \frac{Tz^{19}}{e^{19}} - \frac{56}{60} \times \frac{Vz^{20}}{e^{20}} + \frac{59}{63} \times \frac{Wz^{21}}{e^{21}} - \frac{62}{66} \times \frac{Xz^{22}}{e^{22}} + \&c\,;$$

in which it is evident that the generating fractions of the coefficients of the ſeveral terms are derived from thoſe that immediately precede them by the continual addition of the number 3 to both their numerators and denominators.

An example of the reſolution of a cubick equation by means of the expreſſion $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ *the ſeries* $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} + \frac{Tz^{18}}{e^{18}} + \&c.$ *given in Art.* 5.

ART. 9. Let it be required to reſolve the equation $x^3 - x = \frac{1}{3}$ by means of the ſaid expreſſion.

Here q is $= 1$, and $r = \frac{1}{3}$; and conſequently q^3 is $= 1$, and $\frac{q^3}{27} = \frac{1}{27}$, and $\frac{r}{2} = \frac{1}{6}$, and $\frac{rr}{4} = \frac{1}{36}$, which is leſs than $\frac{1}{27}$, or $\frac{q^3}{27}$. Therefore this equation does not come under CARDAN's rule, but may be reſolved by the expreſſion given in Art. 5, provided that $\frac{rr}{4}$, though leſs than $\frac{q^3}{27}$, is greater than half $\frac{q^3}{27}$, or than $\frac{q^3}{54}$; which it is, becauſe it is $=\frac{1}{36}$, whereas $\frac{q^3}{54}$ is equal only to $\frac{1}{54}$, which is leſs than $\frac{1}{36}$. Therefore the propoſed equation $x^3-x=\frac{1}{3}$ may be reſolved by means of the ſaid expreſſion.

ART. 10. Now, ſince in this caſe q is $=1$, and r is $=\frac{1}{3}$, we ſhall have $\frac{r}{2}$, or e, $=\frac{1}{6}$, and $\frac{q^3}{27}-\frac{rr}{4}$ $(=\frac{1}{27}-\frac{1}{36}=\frac{4}{108}-\frac{3}{108}=\frac{1}{108})=\frac{1}{36\times 3}$; that is, zz will be $=\frac{1}{36\times 3}$, and conſequently z will be $=\frac{1}{6\sqrt{3}}$. Therefore $e+z$ will be $=\frac{1}{6}+\frac{1}{6\sqrt{3}}$ $(=\frac{\sqrt{3}}{6\sqrt{3}}+\frac{1}{6\sqrt{3}}=\frac{3}{6\times 3}+\frac{\sqrt{3}}{6\times 3}=\frac{3+\sqrt{3}}{6\times 3}=\frac{3+\sqrt{3}}{18}=\frac{3+1.732,050,8}{18}=\frac{4.732,050,8}{18})=.262,891,71$; and $e-z$ will be $=\frac{1}{6}-\frac{1}{6\sqrt{3}}$ $(=\frac{3-\sqrt{3}}{18}=\frac{3-1.732,050,8}{18}=\frac{1.267,949,2}{18})=.070,441,62$. Therefore the cube-root of $e+z$ is $=\sqrt[3]{.262,891,71}=.640,607,91$; and the cube-root of $e-z$ is $=\sqrt[3]{.070,441,62}=.412,993,40$; and conſequently $\sqrt[3]{e+z}+\sqrt[3]{e-z}$ is $=.640,607,91+.412,993,40=1.053,601,31$.

ART. 11. It remains that we compute the infinite ſeries $\frac{Czz}{ee}+\frac{Gz^6}{e^6}+\frac{Lz^{10}}{e^{10}}+\frac{Pz^{14}}{e^{14}}+\frac{Tz^{16}}{e^{16}}+$ &c. and extract the cube-root of e, and then multiply the ſaid ſeries into 4 times the ſaid cube-root.

Now the cube-root of e is in this caſe $=\sqrt[3]{\frac{1}{6}}=\frac{1}{\sqrt[3]{6}}=\frac{1}{1.817,121}$; and conſequently $4\sqrt[3]{e}$ is $=\frac{4}{1.817,121}$.

And, ſince zz is $= \frac{1}{36 \times 3}$, and ee is $= \frac{1}{36}$, it follows that $\frac{zz}{ee}$ will be $= \frac{1}{3}$. Therefore $\frac{z^4}{e^4}$ will be $= \frac{1}{9}$, and $\frac{z^6}{e^6} = \frac{1}{27}$, and $\frac{z^{10}}{e^{10}}$ $(= \frac{z^6}{e^6} \times \frac{z^4}{e^4} = \frac{1}{27} \times \frac{1}{9}) = \frac{1}{243}$, and $\frac{z^{14}}{e^{14}}$ $(= \frac{z^{10}}{e^{10}} \times \frac{z^4}{e^4} = \frac{1}{243} \times \frac{1}{9}) = \frac{1}{2187}$. Therefore the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} +$ &c. will in this caſe be $= \frac{C}{3} + \frac{G}{27} + \frac{L}{243} + \frac{P}{2187} +$ &c. $= \frac{.111,111,111, \&c.}{3} + \frac{.023,472,031, \&c.}{27} + \frac{.011,690,017, \&c.}{243} + \frac{.007,414,542, \&c.}{2187} +$ &c. $=$ .037,037,037,&c. + .000,869,334,&c. + 000,048,107, &c. + 000,003,390, &c. + &c. = .037,957,868, &c.

Therefore $4\sqrt[3]{e} \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} +$ &c. is equal to $\frac{4}{1.817,121} \times$ .037,957,868, &c. $= \frac{.151,831,472, \&c.}{1.817,121} =$ .083,556,06.

Conſequently the whole expreſſion $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the ſeries $\frac{Czz}{ee} + \frac{Gz^6}{e^6} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}} +$ &c. is $=$ 1.053,601,31 + .083,556,06 = 1.137,157,37; that is, the root of the propoſed equation $x^3 - x = \frac{1}{3}$ is $=$ 1.137,157,37. Q. E. I.

ART. 12. This value of the root of the equation $x^3 - x = \frac{1}{3}$ is exact to ſix places of figures, the more accurate value of it being 1.137,158,164. We may therefore conclude that the expreſſion here made uſe of to determine the value of x, to wit, $\sqrt[3]{e+z} + \sqrt[3]{e-z} +$

$4\sqrt{}^3e \times$ the infinite ſeries $\frac{Czz}{ee}+\frac{Gz^6}{e^6}+\frac{Lz^{10}}{e^{10}}+\frac{Pz^{14}}{e^{14}}+$ &c. is ſomewhat preferable, with reſpect to the practical reſolution of theſe equations, to the other expreſſion of its value given in the former paper in the Philoſophical Tranſactions, to wit, $\sqrt{}^3e \times$ the infinite ſeries $2+\frac{2zz}{9ce}-\frac{20z^4}{243e^4}+\frac{308z^6}{6561e^6}-$ &c. For it appeared in Art. 42 of that paper, page 941, that the value of the root of this ſame equation $x^3-x=\frac{1}{3}$ obtained by computing four terms of the ſeries $2+\frac{2zz}{9ee}-\frac{20z^4}{243e^4}+\frac{308z^6}{6561e^6}-$ &c. was 1.137,33; which is true only to four places of figures; whereas by computing the ſame number of terms of the ſeries $\frac{Czz}{ee}+\frac{Gz^6}{e^6}+\frac{Lz^{10}}{e^{10}}+\frac{Pz^{14}}{e^{4}}+$ &c. we have juſt now obtained a value of the ſame root, to wit, the number 1.137,157,37, which is exact to ſix places of figures. This is agreeable to what was obſerved above in Art. 6.

A Summary of the Concluſions obtained in this Paper and the former Paper to which it is an Appendix.

ART. 13. I will now conclude this paper by ſetting down, in as conciſe a manner as I can, the ſeveral concluſions that have been obtained in this and the abovementioned paper in the Philoſophical Tranſactions for the year 1778, Number XLII. page 902, &c. concerning the root of the cubick equation $x^3-qx=r$, that the whole may be ſeen together at one view.

ART. 14. If $\frac{rr}{4}$ is greater than $\frac{q^3}{27}$, and e be put $=\frac{r}{2}$, and $ss=\frac{rr}{4}-\frac{q^3}{27}$, it is ſhewn in Art. 5, of the ſaid former paper, that the root of the equation $x^3-qx=r$ will be $=\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}\ \frac{+q}{3\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}}$, or $\sqrt[3]{e+s}+\frac{q}{3\sqrt[3]{e+s}}$.

ART. 15. And it is ſhewn in Art. 9. of the ſaid paper, that the root of the ſaid equation will in that caſe be alſo equal to $\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}\ \frac{q}{3\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}}$, or $\sqrt[3]{e-s}+\frac{q}{3\sqrt[3]{e-s}}$.

ART. 16. And it is ſhewn in Art. 11, of the ſaid paper, page 915, that the root of the ſaid equation will in that caſe be alſo equal to $\sqrt[3]{\frac{r}{2}+\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}+\sqrt[3]{\frac{r}{2}-\sqrt{\frac{rr}{4}-\frac{q^3}{27}}}$, or $\sqrt[3]{e+s}+\sqrt[3]{e-s}$.

ART. 17. And it is ſhewn in Art. 23, of the ſaid paper, page 923, that the root of the ſaid equation will in that caſe be alſo equal to $\sqrt[3]{e}$ × the infinite ſeries

$$2-\frac{2ss}{9ee}-\frac{20s^4}{243e^4}-\frac{308s^6}{6561e^6}-\frac{1870s^8}{59049e^8}-\frac{111,826s^{10}}{4,782,969e^{10}}-\frac{2,358,512s^{12}}{129,140,163e^{12}}-\frac{120,646,960s^{14}}{8,135,830,269e^{14}}$$

$\frac{120,646,960s^{14}}{8,135,830,269e^{14}}$ − &c. or (if we put the capital letters A, B, C, D, E, F, G, H, &c. for the ſeveral numeral coefficients 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{5}{81}$, $\frac{10}{243}$, $\frac{22}{729}$, $\frac{154}{6561}$, $\frac{2618}{137,781}$, &c. of the terms of the ſeries $1+\frac{s}{3e}-\frac{ss}{9ee}+\frac{5s^3}{81e^3}-\frac{10s^4}{243e^4}+\frac{22s^5}{729e^5}-\frac{154s^6}{6561e^6}+\frac{2618s^7}{137,781e^7}-$ &c. which is equal to the cube-root of the binomial quantity $1+\frac{s}{e}$), equal to $\sqrt{}^3 e$ × the infinite ſeries $2A-\frac{2Css}{ee}-\frac{2Es^4}{e^4}-\frac{2Gs^6}{e^6}-\frac{2Is^8}{e^8}-\frac{2Ls^{10}}{e^{10}}-\frac{2N^{12}}{e^{12}}-\frac{2P^{14}}{e^{14}}-$ &c; in which ſeries all the terms after the firſt term 2A, or 2, are marked with the ſign −, or are to be ſubtracted from the ſaid firſt term.

ART. 18. And, if $\frac{rr}{4}$ is leſs than $\frac{q^3}{27}$, but greater than its half, or $\frac{q^3}{54}$, and e be put, as before, $=\frac{r}{2}$, and $zz=\frac{q^3}{27}-\frac{rr}{4}$, it is ſhewn in Art. 31, 32, 33, 34, 35, of the ſaid paper, pages 927—936, that the root of the equation $x^3-qx=r$ will be equal to $\sqrt{}^3 e$ × the infinite ſeries $2+\frac{2zz}{9ee}-\frac{20z^4}{243e^4}+\frac{308z^6}{6561e^6}-\frac{1870z^8}{59049e^8}+\frac{111,826z^{10}}{4,782,969e^{10}}-\frac{2,358,512z^{12}}{129,140,163e^{12}}+\frac{120,646,960z^{14}}{8,135,830,269e^{14}}-$ &c. or $\sqrt{}^3 e$ × the infinite ſeries $2A+\frac{2Czz}{ee}-\frac{2Ez^4}{e^4}+\frac{2Gz^5}{e^6}-\frac{2Iz^8}{e^8}+\frac{2Lz^{10}}{e^{10}}-\frac{2Nz^{12}}{e^{12}}+\frac{2Pz^{14}}{e^{14}}-\frac{2Rz^{16}}{e^{16}}+\frac{2Tz^{18}}{c^{18}}-$ &c; in which ſeries the capital letters A, C, E, G, I, L, N, P, R, T, &c. denote the ſame numeral coefficients 1, $\frac{1}{9}$, $\frac{10}{243}$, $\frac{154}{6561}$, &c. as in the laſt Article, and all the terms that involve the odd powers of the fraction $\frac{zz}{ee}$, to wit, $\frac{zz}{ee}$ itſelf, $\left.\overline{\frac{zz}{ee}}\right|^{3}$, or $\frac{z^{6}}{e^{6}}$, $\left.\overline{\frac{zz}{ee}}\right|^{5}$, or $\frac{z^{10}}{e^{10}}$, $\left.\overline{\frac{zz}{ee}}\right|^{7}$, or $\frac{z^{14}}{e^{14}}$, $\left.\overline{\frac{zz}{ee}}\right|^{9}$, or $\frac{z^{18}}{e^{18}}$, &c. are marked with the ſign +, or are to be added to the firſt term 2A, or 2, and all the other terms are marked with the ſign −, or are to be ſubtracted from the former.

Theſe are the concluſions obtained in the ſaid former paper, which is printed in the Philoſophical Tranſactions for the year 1778, Number XLII. pages 902—949. The concluſions obtained in this paper are as follows.

ART. 19. If $\frac{rr}{4}$ is leſs than $\frac{q^{3}}{27}$, but greater than its half, or $\frac{q^{3}}{54}$, and e be put, as before, $= \frac{r}{2}$, and $zz = \frac{q^{3}}{27} - \frac{rr}{4}$, it is ſhewn in the preſent paper, Art. 5 and 7, that the root of the equation $x^{3}-qx=r$ will be equal to the mixed expreſſion $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the infinite ſeries $\frac{zz}{9ee} + \frac{154z^{6}}{6561e^{6}} + \frac{55913z^{10}}{4,782,969e^{10}} + \frac{60,323,480z^{14}}{8,135,830,269e^{14}}$ + &c. or $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the infinite ſeries .111,111,111, &c. $\times \frac{zz}{ee} +$ .023,472,031, &c. $\times \frac{z^{6}}{e^{6}} +$ .011,690,017, &c. $\times \frac{z^{10}}{e^{10}} +$ .007,414,542, &c. $\times \frac{z^{14}}{e^{14}}$ + &c. or $\sqrt[3]{e+z} + \sqrt[3]{e-z} + 4\sqrt[3]{e} \times$ the infinite ſeries $\frac{Czz}{ee} + \frac{Gz^{6}}{e^{6}} + \frac{Lz^{10}}{e^{10}} + \frac{Pz^{14}}{e^{14}}$ + &c; in which ſeries the capital letters C, G, L, P, &c. denote the ſame numeral coefficients $\frac{1}{9}$, $\frac{154}{6561}$, $\frac{55913}{4,782,969}$, $\frac{60,323,480}{8,135,830,269}$, &c. as they denoted in the two laſt Articles, to wit, the coefficients of $\frac{zz}{ee}$, $\frac{z^6}{e^6}$, $\frac{z^{10}}{e^{10}}$, $\frac{z^{14}}{e^{14}}$, $\frac{z^{18}}{e^{18}}$, &c. or of the odd powers of $\frac{zz}{ee}$ in the ſeries which is equal to $\sqrt[3]{1+\frac{z}{e}}$. And it is alſo ſhewn in Art. 6 and 12 of this paper, that this expreſſion, computed to a given number of terms, gives the value of x ſomewhat more exactly than the former expreſſion, $\sqrt[3]{e} \times$ the infinite ſeries $2+\frac{2zz}{9ee}-\frac{20z^4}{243e^4}+\frac{308z^6}{6561e^6}-$ &c. if computed only to the ſame number of terms.

ART. 20. As to the ſecond branch of the ſecond caſe of the equation $x^3-qx=r$, or that in which $\frac{rr}{4}$ is leſs than half $\frac{q^3}{27}$, or than $\frac{q^3}{54}$, I do not know any method of extending CARDAN's rule to it. But I have been informed by my learned and ingenious friend Dr. CHARLES HUTTON, Profeſſor of Mathematicks in the Royal Academy at Woolwich, that he has diſcovered ſuch a method: and I hope he will ſoon communicate it to this learned Society.